Sensitivity analysis for diffusion processes constrained to an orthant

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Abstract

This paper studies diffusion processes constrained to the positive orthant, and investigates changes in the steady-state distribution of such diffusions under infinitesimal changes in the drift. Our first main result states that any constrained function and its drift-derivative is the unique solution to an augmented Skorohod problem. Our second main results uses this characterization to prove that the steady-state distribution of the joint processes (diffusion and its derivative processes) satisfies a basic adjoint relation. We specialize the technique to the case of reflected Brownian motion.

1 Introduction

This paper is motivated by a desire to better understand the relation between performance metrics and control variables in a network with shared but limited resources. We are specifically interested in service networks, where customers seeking a certain service may suffer from delays as a result of temporary insufficient service capacity. The control variables are the service capacities at the individual stations. Many service processes can be modeled by stochastic (or queueing) networks, and an important question is how resources should be allocated given random fluctuations in the arrivals and its interplay with potentially random service times. Questions of this type are readily answered if the network has a product-form structure Kleinrock (1964); Wein (1989), but few results have been obtained when this assumption fails Dieker et al. (2011); Pollett (2009). It is the goal of this paper to introduce new tools in this context, which could be used in the context of both sensitivity analysis and system optimization.

We study diffusion processes and their 'derivatives', defined as the change in the process under an infinitesimal change in the drift. Although some of our results are stated more generally, this paper focuses on diffusion processes for two reasons. First, this framework allows us to explain key concepts in a tractable yet relatively general setting. Second, diffusion processes are rooted in heavy-traffic approximations for stochastic networks, and the heavy-traffic assumption seems reasonable in the context of resource allocation problems with systems operating close to their capacity. Another feature of this paper is that we

focus on the long-term (steady-state) behavior of diffusions and their derivatives. Although it is certainly desirable to obtain time-dependent tools as well, given the vast body of work on steady-state results, making this assumption is a natural first step. The techniques developed in this paper are likely to be also relevant in the time-dependent case.

We have two main results. The first is a statement on the behavior of deterministic functions under the well-known Skorohod reflection map with oblique reflection (regulation), and states that the map and its 'derivative' are the unique solution to an augmented version of the Skorohod problem. Our proof of this result relies on recent insights into directional derivatives by Mandelbaum and Ramanan (2010), which have been developed in the context of time-inhomogeneous systems but are shown here to be useful for sensitivity analysis as well.

For our second main result, we specialize to diffusion processes and study the steady-state distribution of the diffusion process jointly with its derivative processes. Specifically, given a constrained diffusion process Z representing the dynamics of the underlying stochastic network (i.e., the queue lengths at each of the stations), we let the stochastic process A represent the change in Z under an infinitesimal change in the drift. This process has jumps even if Z is continuous. Our result is that the steady-state distribution of the joint processes (Z, A) satisfies a kind of basic adjoint relation, which is the analog of the equation $\pi' Q = 0$ for continuous-time Markov processes on a discrete state space. The proof relies on a delicate analysis of the jumps of A.

The intuition behind the program carried out in this paper can be summarized as follows. Suppose Z^ϵ is a constrained diffusion process with drift coefficient $\mu(\cdot) - \epsilon v$ in the interior of the orthant, where v is an arbitrary nonnegative vector. Suppose the processes $\{Z^\epsilon\}$ are driven by the same Brownian motion for every $\epsilon \geq 0$, so that they are coupled. The processes $Z \equiv Z^0$ and Z^ϵ are Markovian, and one can therefore expect to be able to give a basic adjoint relationship for their stationary distributions (if they exist). Moreover, (Z, Z^ϵ) and therefore $(Z, (Z^\epsilon - Z)/\epsilon)$ can be expected to be Markovian as a result of the coupling. Provided one can make sense of the limit as $\epsilon \to 0+$, one can expect that the distribution of the limit satisfies a similar relationship. This results in an 'augmented' basic adjoint relationship, which we state in Theorem 3. The constrained diffusion processes studied in this paper are reflected Brownian motions (RBMs). Using standard tools, our results can be formulated for the much larger class of constrained diffusion processes such as pathwise solutions to stochastic differential equations with reflection Dupuis and Ishii (1991); Ramanan (2006), but we focus on RBM to be able to highlight the main ideas.

When carrying out the aforementioned approach for reflected Brownian motion, we were surprised to find that the natural analog of the one-dimensional basic adjoint relationship (or the product-form solution with normal reflection) fails to characterize the stationary distribution of (Z, A). In fact, even though Z is known not to spend any time on low-dimensional faces, it is critical to incorporate the jumps of A when Z reaches those faces in order to formulate the basic adjoint relationship.

This work has the potential to lead to new numerical methods in the context of opti-

mization and sensitivity analysis for queueing networks, which remove the need for computationally intensive or numerically unstable operations such as gradient estimation. To explain, due to the division by ϵ, any performance metric of $(Z^0 - Z^\epsilon)/\epsilon$ suffers from numerical instability issues for small $\epsilon > 0$. Researchers in stochastic optimization have developed several techniques to mitigate this effect (see, e.g., Asmussen and Glynn (2007)). The approach taken in this paper is to analytically describe and investigate the dynamics of the limit. Even for networks not in heavy traffic, this could lead to new bias reduction methods in the context of stochastic optimization.

The framework of this paper is related to a body of literature known as infinitesimal perturbation analysis Glasserman (1991, 1993, 1994); Heidergott (2006). Infinite perturbation analysis also aims to perform sensitivity analysis or gradient estimation for performance metrics in (say) a queueing network, and it does so by formulating conditions under which an expectation and a derivative operator can be interchanged. Here, however, we do not seek such an interchange involving a performance metric but instead we study the (whole) steady-state distribution of a stochastic process with its derivative process.

This paper is outlined as follows. Section 2 summarizes our approach in the one-dimensional case, which serves as a guide for our multi-dimensional results. Section 3 discusses two technical preliminaries: oblique reflection maps and their derivatives. In Section 4 we formulate our two main results. Section 5 is devoted to the proof of the first main result, while Section 6 gives the proof of the second main result. A key role is played by jump measures, for which we obtain a description in Section 7. The appendices contain several technical digressions.

Notation

For $J \in \mathbb{N}$, $\mathbb{R}^J$ denotes the J-dimensional Euclidean space. We denote the space of real $n \times m$ matrices by $\mathbb{M}^{n\times m}$, and the subset of nonnegative matrices by $\mathbb{M}_+^{n\times m}$. All vectors are to be interpreted as column vectors, and we write M^j and M_i for the j-th column and the i-th row of a matrix M, respectively. In particular, v_i is the i-th element of a vector v and M_i^j is element (i,j) of a matrix M. Similarly, given a set $I \subseteq \{1,\dots,J\}$, we write M_I and M^I for the matrices consisting of the rows and columns of M, respectively, with indices in I. Throughout, E stands for the identity matrix and we write δ_i^j for E_i^j. We use the symbol $'$ for transpose.

Given a space S, a vector-valued function $h : S \to \mathbb{R}^J$ on S, and a vector of measures $\nu = (\nu_1, \dots \nu_J)$ on S, we set

$$\int h(x)\nu(dx) = \int h(x) \cdot \nu(dx),$$

provided the right-hand side exists. We shall also employ this notation when h is matrix-valued. That is, we write for $h : S \to \mathbb{M}^{J\times J}$,

$$\int h(x)\nu(dx) = \int \langle h(x), \nu(dx)\rangle_{\mathrm{HS}},$$

where $\langle \cdot, \cdot \rangle_{\text{HS}}$ is the Hilbert-Schmidt inner product on $\mathbb{M}^{J\times J}$ given by

$$\langle M_1, M_2 \rangle_{\text{HS}} = \operatorname{tr}(M_1' M_2).$$

For a function $g : \mathbb{M}^{J\times J} \to \mathbb{R}$, we define $\nabla g : \mathbb{M}^{J\times J} \to \mathbb{M}^{J\times J}$ as the function for which element (i, j) is given by the directional derivative of g in the direction of the matrix with only zero entries except for element (i, j), where its entry is 1. We also write, for $i = 1, \ldots, J$, $F_i = \{(z, a) \in \mathbb{R}_+^J \times \mathbb{M}^{J\times J} : z_i = 0\}$, $F_i^a = \{(z, a) \in \mathbb{R}_+^J \times \mathbb{M}^{J\times J} : a_i = 0\}$. The space of functions $f : \mathbb{R}_+^J \times \mathbb{M}_+^{J\times J} \to \mathbb{R}$ which are twice continuously differentiable with bounded derivatives is denoted by $C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$.

We write $\mathbb{D}_+^J$ for the space of $\mathbb{R}_+^J$-valued functions on $\mathbb{R}_+$ which are right-continuous on $\mathbb{R}_+$ with left limits in $(0, \infty)$. The subset of continuous functions is written as C^J, and C_+^J denotes the set of nonnegative continuous functions. Similarly, we write $\mathbb{D}^{J\times J}$ for the space of $\mathbb{M}^{J\times J}$-valued right-continuous functions on $\mathbb{R}_+$ with left limits. The subset of $\mathbb{M}_+^{J\times J}$-valued functions is denoted by $\mathbb{D}_+^{J\times J}$.

2 A motivating one-dimensional result

Let Z be a one-dimensional reflected Brownian motion with drift $\theta < 0$ and variance σ^2. That is,

$$Z(t) = X(t) + Y(t) \geq 0,$$

where $X(t)$ is Brownian motion with drift θ and variance σ^2, and the regulating term Y is given by

$$Y(t) = \max\left(\sup_{0\leq s\leq t} [-X(s)], 0 \right).$$

Similarly, for $\epsilon > 0$, let Z^ϵ be one-dimensional Brownian motion with drift $\theta - \epsilon$ and variance σ^2. It follows from Mandelbaum and Massey (1995) that

$$A(t) \triangleq \lim_{\epsilon\to 0+} \frac{1}{\epsilon}(Z(t) - Z^\epsilon(t)) \tag{2.1}$$

exists as $\epsilon \to 0+$. Moreover, we have an explicit formula:

$$A(t) = t - B(t), \tag{2.2}$$

where

$$B(t) = \sup\{s \in [0, t] : Z(s) = 0\},$$

and $\sup \emptyset = 0$ by convention. In view of the definition of A in (2.1), we call it the derivative process of Z.

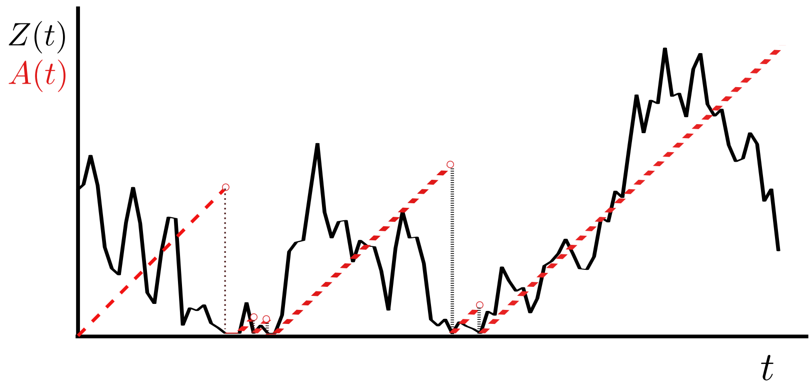

Figure 1: Sample paths of (Z, A) as a function of time. The solid black curve is Z, while the dashed red curve is A. The slope of A is 1 whenever it is continuous, and A jumps to 0 whenever Z hits 0.

We now relate these notions to sensitivity analysis. Making the dependence on θ explicit, our investigations are motivated by the following sequence of equalities: for any 'smooth' function (performance measure) ϕ,

$$\frac{d}{d\theta}E\left[\phi(Z^\theta(\infty))\right] = E\left[\frac{d}{d\theta}\phi(Z^\theta(\infty))\right] = E\left[A^\theta(\infty)\phi'(Z^\theta(\infty))\right]. \tag{2.3}$$

Thus, to study (infinitesimal) changes in the steady-state performance measure under infinitesimal changes in the drift θ, one is led to investigating the stationary distribution of (Z, A) (assuming it exists). We are indeed able to justify the above equalities in the one-dimensional case.

One readily checks that the sample paths of the process B are nondecreasing, that they are left-continuous with right-hand limits, and that A has jumps. In particular, the process A is of finite variation and (Z, A) is a semimartingale with jumps. An illustration of the process (Z, A) is given in Figure 1. From Ito's formula in conjunction with sample path properties of A, we obtain the following result. We suppress further details of the proof, since this program is carried out in greater generality in the proof of Theorem 3.

Theorem 1. *Let Z be a one-dimensional reflected Brownian motion with drift θ and variance σ^2. Let A be defined in (2.2). Suppose that the process (Z, A) has a unique stationary distribution π. For any $f \in C_b^2(\mathbb{R}_+ \times \mathbb{R}_+)$, we have the following relationship:*

$$\begin{aligned} 0 \quad = \quad & \int_0^\infty \int_0^\infty \left[\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial z^2} f(z, a) + \theta \frac{\partial}{\partial z} f(z, a) + \frac{\partial}{\partial a} f(z, a) - \frac{\partial}{\partial a} f(0, a)\right] \pi(dz, da) \\ & - \frac{\partial}{\partial z} f(0, 0)\theta. \end{aligned} \tag{2.4}$$

One can go further and derive the Laplace transform of π using this theorem, see Appendix A. One then finds that, for any $\alpha, \eta > 0$,

$$\int_0^\infty \int_0^\infty e^{-\alpha z - \eta a} \pi(dz, da) = \frac{-2\theta}{\alpha\sigma^2 - \theta + \sqrt{2\eta\sigma^2 + \theta^2}}. \tag{2.5}$$

In particular, the theorem completely determines the stationary measure π. It is also possible to derive this result immediately from standard fluctuation identities for Brownian motion with drift, using results from Dębicki et al. (2007). In fact, since the corresponding densities are known explicitly (or can be found by inverting the Laplace transform), it is possible to write down the density of $(Z(\infty), A(\infty))$ in closed form. Using the resulting expression, it can be verified directly that (2.3) indeed holds.

3 Oblique reflection maps and their directional derivatives

This section contains the technical preliminaries to formulate a higher-dimensional analog of Theorem 1. To introduce the analogs of the derivatives a and b, we need the following definition.

Definition 1. *(Oblique reflection map) Suppose a given $J \times J$ real matrix R can be written as $R = I - P$, where P is a nonnegative matrix with spectral radius less than one. Then for every $x \in \mathbb{D}^J$, there exists a unique pair $(y, z) \in \mathbb{D}_+^J \times \mathbb{D}_+^J$ satisfying the following conditions:*

1. *$z(t) = x(t) + Ry(t) \geq 0$ for $t \geq 0$,*
2. *$y(0) = 0$, y is componentwise nondecreasing and*

$$\int_0^\infty z(t) dy(t) = 0.$$

We write $y = \Phi(x)$ and $z = \Gamma(x)$ for the oblique reflection map.

The reflection map gives rise to derivatives as formalized in the following definition. Existence of the derivatives is guaranteed by Theorem 1.1 in Mandelbaum and Ramanan (2010).

Definition 2. *(Derivatives of the reflection map) Let $\chi(t) = tE$ and define the $\mathbb{M}^{J\times J}$-valued functions a and b by setting $a = \lim_{\epsilon\to 0+} a_\epsilon$ and $b = \lim_{\epsilon\to 0+} b_\epsilon$, where, for $j = 1, \ldots, J$,*

$$\begin{aligned} a_\epsilon^j &\triangleq \frac{1}{\varepsilon}\left[\Gamma(x) - \Gamma(x - \epsilon\chi^j)\right], \\ b_\epsilon^j &\triangleq -\frac{1}{\varepsilon}\left[\Phi(x) - \Phi(x - \epsilon\chi^j)\right]. \end{aligned} \tag{3.1}$$

Then we have for each $t \geq 0$,

$$a(t) = tE - Rb(t). \tag{3.2}$$

For notational convenience, we write $a = \Gamma'(x)$ and $b = -\Phi'(x)$.

4 Main results

This section states the main results of this paper. The first result makes the connection between derivatives and an augmented Skorohod problem, which we define momentarily. The second result is a basic adjoint relationship for the steady-state distribution of reflected Brownian motion and its derivative process, which is the analog of the equation $\pi' Q = 0$ mentioned in the introduction.

4.1 Augmented Skorohod problems and derivatives

In this section we introduce the augmented Skorohod problem and connect it with derivatives of the oblique reflection map.

Definition 3. *(Augmented Skorohod problem) Suppose we are given two $J \times J$ real matrices $R = I - P$ and $\tilde{R} = I - \tilde{P}$, where both P and $\tilde{P}$ are nonnegative matrices with spectral radius less than one. Given $(x, \chi) \in C^J \times C^J$ with χ nonnegative and nondecreasing, we say that $(z, y, a, b) \in C_+^J \times C_+^J \times \mathbb{D}_+^{J \times J} \times \mathbb{D}_+^{J \times J}$ satisfies the augmented Skorohod problem associated with $(R, \tilde{R})$ for (x, χ) if the following conditions are satisfied:*

1. *$z(t) = x(t) + Ry(t)$ for $t \geq 0$,*
2. *$y(0) = 0$, y is componentwise nondecreasing and*
$$\int_0^\infty z(t) dy(t) = 0.$$
3. *$a(t) = \chi(t) - \tilde{R}b(t)$ for $t \geq 0$,*
4. *$b(0) = 0$, $b(t) \geq 0$, b is componentwise nondecreasing and, for $j = 1, \ldots, J$,*
$$\int_0^\infty z(t) db^j(t) = 0. \tag{4.1}$$
5. *For $i = 1, \ldots, J$ and $t \geq 0$, $z_i(t) = 0$ implies $a_i(t) = 0$.*

Building on results from Mandelbaum and Ramanan (2010), we show in Appendix B that the augmented Skorohod problem has a unique solution. To interpret solutions to the augmented Skorohod problem, we found it easiest to think of the dynamics of (z, a^j) for

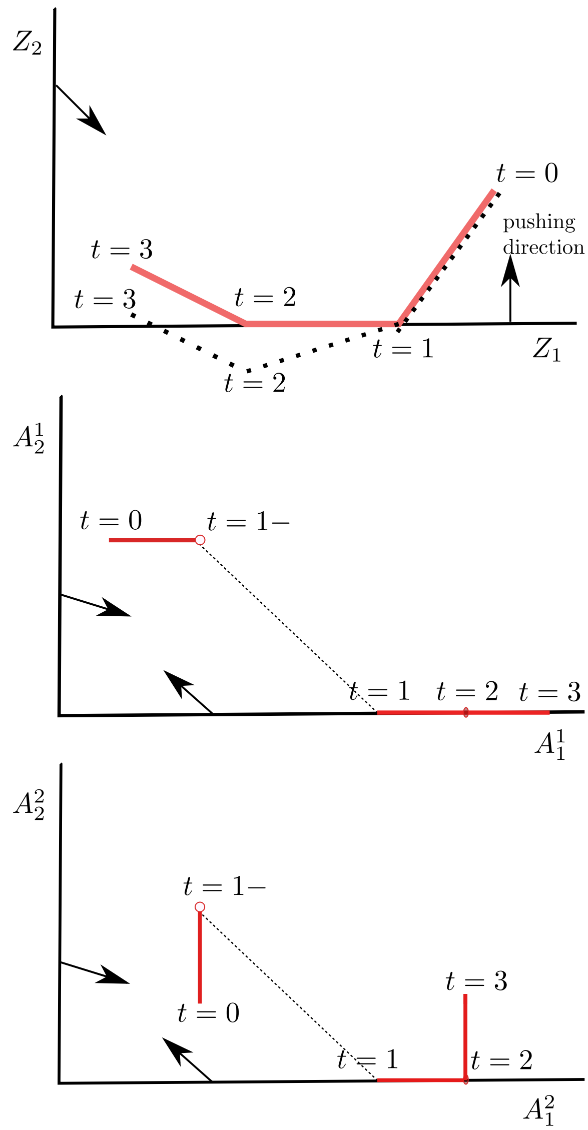

Figure 2: The first diagram depicts a trajectory of Z, with corresponding 'free' path X (dotted). In the second and third diagram, the trajectories of A^1 and A^2 travel at unit rate right and up, respectively, until Z hits $\partial\mathbb{R}_+^2$. The face $Z_2 = 0$ is hit at time $t = 1$, causing A^1 and A^2 to jump to the faces $A_2^1 = 0$ and $A_2^2 = 0$, respectively, in direction $\tilde{R}^2$.

each $j = 1, \ldots, J$ separately. When z hits the face $z_I = 0$, then a^j jumps to the face $a^j_I = 0$ in the direction of the unique vector in the column space of $\tilde{R}^I$ which brings it to that face. We refer to Figure 2 for an illustrative example in the two-dimensional case.

Unlike requirements 2 and 4 in Definition 3, requirement 5 is not a 'complementary' condition. In view of the sample path dynamics in Figure 2, it may seem reasonable to replace requirement 5 by $\int_0^\infty a^j(t)dy(t) = 0$ or another complementarity condition between (y, z) and (a, b). In that case, however, the augmented Skorohod will fail to have a unique solution. This can be seen by verifying that both the left-derivative and the right-derivative of the reflection map satisfy $\int_0^\infty a^j(t)dy(t) = 0$ but only the left-derivative (as defined in Definition 3) satisfies requirement 5.

We now make a connection between derivatives (sensitivity analysis) and solutions to the augmented Skorohod problem.

Theorem 2. *Fix some $x \in C^J$, and let $z = \Gamma(x)$ and $y = \Phi(x)$ be given by the oblique reflection map. Define the derivatives $a = \Gamma'(x)$ and $b = -\Phi'(x)$ as in Definition 2. Set $\chi(t) = tE$ for $t \geq 0$. Then (z, y, a, b) satisfies the augmented Skorohod problem associated with (R, R) for (x, χ).*

4.2 Steady-state distribution of RBM and its derivative process

Our second main result shows that the stationary distribution of reflected Brownian motion jointly with its derivative process satisfies an extension of the basic adjoint relationship (BAR) for reflected Brownian motion. The proof relies on Ito's formula in conjunction with properties developed in the previous section, and the result can be formulated for a much larger class of constrained diffusion processes with continuous sample paths such as strong solutions to SDERs. To be able to focus on the critical ideas, we only present the proof for reflected Brownian motion.

We start with some definitions. When X is a J-dimensional Brownian motion with $X(0) = x$ with drift θ and covariance matrix Σ, $Z = \Gamma(X)$ is a so-called reflected Brownian motion. The process Y with sample paths $Y = \Phi(X)$ is called the regulator of X. We note that it is not *always* possible to define reflected Brownian motion through a pathwise mapping, see for instance Dieker (2010) for details and references. Throughout this paper, we only work with reflected Brownian motions obtainable through the oblique reflection map of Definition 1, but extensions of our result are presumably possible.

We define the following operators: for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$ and each set $I \subseteq \{1, 2, \ldots, J\}$, let

$$(O_I f)(z, a) = \sum_{S \subseteq \{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, Q_I(a)), \tag{4.2}$$

$$O = \sum_{I \subseteq \{1,\ldots,J\}} O_I, \tag{4.3}$$

where $\Pi_{S\cup I}$ is the projection operator which sets the coordinates in $S \cup I$ equal to 0, and Q_I is also an projection operator with the following property. The matrix $Q_I(a)$ is obtained from a by subtracting columns of R^I, in such a way that the rows of $Q_I(a)$ with indices in I become zero. Explicitly, we have

$$Q_I(a) = a - R^I(R_I^I)^{-1}a_I, \tag{4.4}$$

where R_I^I is the principal submatrix of R obtained by removing rows and columns from R which do not lie in I. When $I = \emptyset$, we set $Q_I(a) = a$ for $a \in \mathbb{M}^{J\times J}$.

We also define operators L and T on $C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ as follows:

$$\begin{aligned} L &= \frac{1}{2}\sum_{i,j=1}^{J} \Sigma_{ij}\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{j=1}^{J}\theta_j \frac{\partial}{\partial z_j}, \\ T &= L + \mathrm{tr}(\nabla_a), \end{aligned} \tag{4.5}$$

where we recall that, in view of our notational conventions, $\mathrm{tr}(\nabla_a)$ is shorthand for $\sum_{i,j=1}^{J} \frac{d}{da_{ij}}$.

We can now formulate the following theorem, which is our second main result.

Theorem 3. *Let X be a Brownian motion with drift θ and covariance matrix Σ. Let $Z = \Gamma(X)$ be the resulting reflected Brownian motion and $A = \Gamma'(X)$ be the derivative of Z as in Definition 2. Suppose (Z, A) has a unique stationary distribution π. Then there exists finite Borel measure ν such that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$, the following relationship holds:*

$$\int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} [T \circ Of](z,a)d\pi(z,a) + \int_{\bigcup_i (F_i \cap F_i^a)} [R'\nabla_z(Of)(z,a)]d\nu(z,a) = 0, \tag{4.6}$$

where the operators O and T are given in (4.3) and (4.5), and the Laplace transform of ν is given in (6.6) below.

We remark that the proof of the theorem shows that (4.6) is equivalent to several equations. That is, for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ and each set $I \subseteq \{1, 2, \ldots, J\}$, π must satisfy

$$\int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} [T \circ O_I f](z,a)d\pi(z,a) + \int_{\bigcup_i (F_i \cap F_i^a)} [R'\nabla_z(O_I f)(z,a)]d\nu(z,a) = 0, \tag{4.7}$$

where the operators O_I are defined in (4.2). Note that (4.7) produces 2^J equations, one of which is trivial.

It is natural to ask whether (4.6) or (4.7) fully determines the stationary distribution π. It is outside the scope of this paper to attempt to prove this, but we give some supporting evidence by showing that: 1) restricting to functions only depending on z and not on a

yields the classical BAR, 2) in the one-dimensional case, we recover the relationship from Section 2, 3) if $R = 0$ (normal reflection directions), π factorizes as a product of J measures ('product-form'), 4) varying $\tilde{R}$ leads to a different equation.

We first check that (4.6) yields the classical BAR for the stationary distribution of the reflected Brownnian motion Z when choosing $f(z, a) \equiv g(z)$. One readily checks that in this case,

$$(Of)(z, a) = \sum_{I \subseteq \{1,2,\ldots,J\}} \sum_{S \subseteq \{1,\ldots,J\} \setminus I} (-1)^{|S|} g(\Pi_{S \cup I} z) = g(z) - g(0).$$

Substituting the above equation in (4.6), we immediately obtain the well-known basic adjoint relationship Harrison and Williams (1987a):

$$\int_{\mathbb{R}_+^J} Lg(z) d\pi(z) + \int_{\bigcup_i F_i} [R' \nabla_z g(z)] d\nu(z) = 0, \tag{4.8}$$

where $d\pi(z) = \int_{a \in \mathbb{M}^{J \times J}} d\pi(z, a)$ is the stationary distribution for Z and the Borel measure $d\nu(z)$ is given by $d\nu(z) = \int_{a \in \mathbb{M}^{J \times J}} d\nu(z, a)$.

We next check that (4.6) recovers Theorem 1, which shows in particular that (4.6) fully determines π in the one-dimensional case, i.e., for $J = 1$. Indeed, it is readily seen that

$$(Of)(z, a) = (O_{\emptyset} f)(z, a) + (O_{\{1\}} f)(z, a) = f(z, a) - f(0, a).$$

Combining this with (4.6) gives (2.4).

Finally, we verify that (4.6) determines π for $J \geq 2$ if R is identity matrix, i.e., $P = 0$. In this case, we have by definition that $A_{ij} \equiv 0$ for $i \neq j$. To determine the joint stationary distribution of (Z, A), which is of product form, it suffices to show that (4.6) or (4.7) yields the correct marginal distributions $\pi(dz_j, da_{jj})$ for each $j = 1, 2, \ldots, J$. Let $h \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$. If we set $f(z, a) \equiv h(z_j, a_{jj})$ and $I = \{1, 2, \ldots, J\} \setminus \{j\}$ in (4.7), we deduce that

$$(O_I f)(z, a) = h(z_i, a_{jj}) - h(0, a_{jj}).$$

Substituting this into (4.7), and using the fact that total mass of ν is $-\theta$ when Z is of one dimension, we immediately get the one-dimensional result (2.4), which characterizes the marginal distribution $\pi(dz_j, da_{jj})$. Therefore, (4.6) also determines π in this case.

We next give an informal argument why (4.7) for $I = \emptyset$ cannot be sufficient to determine π, even though the equation involves an integral of π over the interior and all low-dimensional faces and this equation determines π completely in the one-dimensional case. Theorem 3 gives the basic adjoint relationship for the augmented Skorohod problem associated with (R, R) for (X, χ), but a similar equation can be given with (R, R) replaced by $(R, \tilde{R})$. This would lead to the same equations (4.7), but the definition of Q_I in (4.4) would involve the matrix $\tilde{R}$ instead of R. For $I = \emptyset$, however, this equation becomes independent of $\tilde{R}$. One expects π to depend on the choice of $\tilde{R}$, in which case (4.7) for $I = \emptyset$ cannot determine π.

5 Characteristics of derivatives and proof of Theorem 2

In this section, we prove Theorem 2. We also collect additional sample path properties of derivatives, with an emphasis on their jump behavior. These properties will be used in the proof of Theorem 3.

Throughout this section, we work under the conditions of Theorem 2. That is, we assume that $x \in C^J$ is given and we write $z = \Gamma(x)$, $y = \Phi(x)$, $a = \Gamma'(x)$ and $b = -\Phi'(x)$. We also set $\chi(t) = tE$ for $t \geq 0$.

5.1 Complementarity

This section connects the augmented Skorohod problem associated with (R, R) for (X, χ) with (z, a). Note that, in view of Definitions 1 and 2, the first two requirements of the augmented Skorohod problem in Definition 3 are immediately satisfied for (x, y, z). It is immediate that $a = \chi - Rb$ by definition of a, so we must indeed choose $\tilde{R} = R$. We proceed with showing that a and b lie in $\mathbb{D}_+^{J\times J}$ as required for the augmented Skorohod problem, but it is convenient to first establish part of the fourth requirement.

Lemma 1. *The $\mathbb{M}^{J\times J}$-valued function b is componentwise nonnegative and nondecreasing.*

Proof. Since $\chi(t) = tE$ for $t \geq 0$, χ is evidently nonnegative and nondecreasing. The monotonicity result in Theorem 6 of Kella and Whitt (1996) shows that for any fixed $\epsilon > 0$, each component of b_ϵ is nonnegative and nondecreasing. The lemma follows from the fact that b is the pointwise limit of the sequences $\{b_\epsilon\}$ as $\epsilon \to 0+$. $\square$

Lemma 2. *The $\mathbb{M}^{J\times J}$-valued functions a and b lie in $\mathbb{D}_+^{J\times J}$.*

Proof. Since b is nonnegative in view of Lemma 1, we have shown the claim for b after verifying that $b \in \mathbb{D}^{J\times J}$. We deduce from Theorem 1.1 in Mandelbaum and Ramanan (2010) that each component of b is upper semicontinuous and that it has left and right limits everywhere. Since b is nondecreasing by Lemma 1, these properties imply that $b \in \mathbb{D}^{J\times J}$.

We next show that $a \in \mathbb{D}_+^{J\times J}$. Clearly, since $b \in \mathbb{D}_+^{J\times J}$, we only need to show that a is nonnegative. Again by the monotonicity result in Theorem 6 of Kella and Whitt (1996), for any fixed $\epsilon > 0$, each component of a_ϵ is nonnegative. This completes the proof of the lemma after letting $\epsilon \to 0+$. $\square$

We next investigate the fourth and fifth requirement of Definition 3. To this end, we need a characterization of b which relies heavily on Mandelbaum and Ramanan (2010).

Lemma 3. *b is the unique solution to the following system of equations: for $i, j = 1, \ldots, J$, and $t \geq 0$,*

$$b_i^j(t) = \sup_{s \in \Phi_{(i)}(t)} [\delta_i^j s + [P' b^j]_i(s)],$$

where the supremum over an empty set should be interpreted as zero and

$$\Phi_{(i)}(t) \quad = \quad \{s \in [0,t] : z_i(s) = 0\}. \tag{5.1}$$

Proof. We use Theorem 1.1 of Mandelbaum and Ramanan (2010), which can be simplified in view of Lemma 1 and the nonnegativity of the matrix P. This theorem states that

$$b_i^j(t) = \begin{cases} 0 & \text{if } t \in (0, t_{(i)}), \\ \sup_{s \in \Psi_{(i)}(t)} [\delta_i^j s + [P' b^j]_i(s)] & \text{if } t \in [t_{(i)}, \infty), \end{cases} \tag{5.2}$$

where $t_{(i)} = \inf\{t \geq 0 : Z_i(t) = 0\}$ and $\Psi_{(i)}(t) = \{s \in [0,t] : Z_i(s) = 0, Y_i(s) = Y_i(t)\}$. Observe that, again using Lemma 1, the supremum must be attained at the rightmost end of the closed interval $\Psi_{(i)}(t)$. Since Y is nondecreasing, this is also the rightmost point of the closed set $\Phi_{(i)}(t)$. This establishes the lemma in view of the convention used for the supremum of an empty set. □

Lemma 4. *Fix any $j = 1, \ldots, J$, we have*

$$\int_0^\infty z(t) db^j(t) = 0. \tag{5.3}$$

Proof. Fix some $i = 1, \ldots, J$. Note that if $z_i(t) > 0$ at time t, we deduce from the path continuity of z that there exists some $\epsilon > 0$ such that $z_i(s) > 0$ for $s \in (t - \epsilon, t + \epsilon)$. This implies that $\Phi_{(i)}(s)$ is constant as a set-valued function for $s \in (t - \epsilon, t + \epsilon)$. Thus $b_i(s)$ is constant for $s \in (t - \epsilon, t + \epsilon)$ by (5.2). Since i is arbitrary, this yields (5.3). □

Lemma 5. *If $z_i(t) = 0$ for some i, then we have $a_i(t) = 0$.*

Proof. Suppose $z_i(t) = 0$. In view of Lemma 1, we deduce from (5.2) that, for any $j = 1, \ldots, J$,

$$b_i^j(t) = \delta_i^j t + [P' b^j]_i(t).$$

Now it follows from (3.2) and $R = I - P'$ that

$$a_i^j(t) = \delta_i^j t - [R b^j]_i(t) = \delta_i^j t - \gamma_i^j(t) + [P' b^j]_i(t) = 0,$$

which completes the proof of the lemma. □

The above two lemmas together with Lemma 2 yield two further complementary conditions.

Corollary 1. *For any $j = 1, \ldots, J$, we have*

$$\begin{aligned} \int^\infty a^j(t) dy(t) &= 0, \\ \int_0^\infty a^j(t) db^j(t) &= 0. \end{aligned} \tag{5.4}$$

Proof of Theorem 2. The claim is now immediate from (3.1) in conjunction with Lemmas 1, 2, 4, and 5. □

5.2 Jumps of a

In this section, we collect sample path properties of a related to its jump behavior. This plays a critical role in the derivation of Theorem 3.

The next lemma states that a is linear whenever z is in the interior of $\mathbb{R}^J_+$.

Lemma 6. *If $z(t) \in \mathbb{R}^J_+ \backslash \partial\mathbb{R}^J_+$ for $t \in [\alpha, \beta]$, then we have for $t \in [\alpha, \beta]$*

$$a(t) = a(\alpha) + (t - \alpha)I.$$

In particular, a is continuous on (α, β) and can only have jumps when $z \in \partial\mathbb{R}^J_+$.

Proof. In view of (3.2), it suffices to show that b is constant for $t \in [\alpha, \beta]$. Since $z(t) \in \mathbb{R}^J_+ \backslash \partial\mathbb{R}^J_+$ for $t \in [\alpha, \beta]$, we obtain from (5.1) that for each $i = 1, \ldots, J$, $\Phi_{(i)}(t)$ is constant as a set-valued function. Therefore, we deduce from (5.2) that $b(t)$ is a constant in $\mathbb{M}^{J\times J}$ for $t \in [\alpha, \beta]$. The proof of the lemma is complete. □

For any function g on $\mathbb{R}_+$, we write $\Delta g(t) = g(t) - g(t-)$. In view of the above lemma, we can characterize the continuous part of the function a. Formally, we write

$$a(t) = a^c(t) + a^d(t),$$

where

$$a^d(t) = \sum_{s \le t} \Delta a(s).$$

We have the following corollary.

Corollary 2. *$a^c(t) = a^c(0) + tE$ for any $t \ge 0$.*

We next characterize the jump direction of a when a jump occurs.

Lemma 7. *Fix a nonempty set $I \subseteq \{1, 2, \ldots, J\}$ and some $t > 0$. Suppose that $z_k(t) = 0$ for $k \in I$ and $z_i(t) > 0$ for $i \notin I$. If $\Delta a(t) \ne 0$, then we must have*

$$\Delta a(t) = -\sum_{k \in I} R^k [\Delta b]_k(t).$$

Proof. Since $z_i(t) > 0$ for $i \notin I$, we deduce from the sample path continuity of z that there exists some $\epsilon > 0$ such that for $i \notin I$, $z_i(s) > 0$ for $s \in (t - \epsilon, t]$. This yields that for $i \notin I$, $\Phi_{(i)}(s)$ is a constant as a set-valued function for $s \in (t - \epsilon, t]$. From (5.2) we infer that for $i \notin I$, $b_i(s)$ is constant for $s \in (t - \epsilon, t]$. This implies that $[\Delta b]_i(t) = 0$ for $i \notin I$, and therefore that

$$\Delta a(t) = -R\Delta b(t) = -\sum_{k=1}^{J} R^k [\Delta b]_k(t) = -\sum_{k \in I} R^k [\Delta b]_k(t).$$

This completes the proof of the lemma. □

6 A basic adjoint relationship and proof of Theorem 3

This section is devoted to the proof of Theorem 3. The key idea is to apply Ito's formula to the semimartingale (Z, A) and use sample path properties of (Z, A) to analyze the stationary measure. Although this is a standard approach in the context of reflected Brownian motion, the analysis here is delicate due to the presence of jumps in the process A.

6.1 Ito's formula for the semimartingale (Z, A)

In this section, we apply Ito's formula to the semimartingale (Z, A).

We first show that (Z, A) is a semimartingale, i.e, each of its component is a semimartingale. Loosely speaking, a semimartingale is an adapted process which is the sum of a local martingale and a finite variation process, with sample paths in $\mathbb{D}$. For more details, we refer readers to (Protter, 2005, Ch. 3) or (Jacod and Shiryaev, 2003, Ch. 1).

Lemma 8. (Z, A) *is a semimartingale.*

Proof. Clearly, (Z, A) is a process adapted to the natural filtration of X. We know from Lemma 2 that each component of the process (Z, A) lies in $\mathbb{D}$. Since Z is a semimartingale, to show (Z, A) is a semimartingale, it suffices to show that A is a semimartingale. In fact, from Lemma 1 and (3.2) we immediately deduce that A is a finite variation process, that is, the paths of A are almost surely of finite variation on $[0, T]$ for any $T > 0$. In particular, A is a semimartingale. $\square$

Since Z has drift θ and covariance matrix Σ, we can rewrite Z as follows: for each $t \geq 0$,

$$Z(t) = Z(0) + \theta t + W(t) + RY(t),$$

where $W(t)$ is a driftless Brownian motion with covariance matrix Σ.

By Ito's formula, e.g., (Jacod and Shiryaev, 2003, Sec. I.4), for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$, we have

$$\begin{aligned} f(Z(t), A(t)) \quad = \quad & f(Z(0), A(0)) + \int_0^t \nabla_z f(Z(s), A(s-)) dW(s) \\ & + \int_0^t [R' \nabla_z f(Z(s), A(s-))] dY(s) \\ & + \int_0^t Lf(Z(s), A(s-)) ds + \int_0^t \nabla_a f(Z(s), A(s-)) dA^c(s) \\ & + \sum_{s \leq t} [f(Z(s), A(s)) - f(Z(s), A(s-))]. \end{aligned} \tag{6.1}$$

(Note that, compared to the formulation in Theorem I.4.57 of Jacod and Shiryaev (2003), we have absorbed the last sum of the jump part into the integral $\int_0^t \nabla_a f(Z(s), A(s-)) dA^c(s)$.)

Suppose that, under appropriate conditions, (Z, A) is positive recurrent and has a unique stationary distribution π. After taking expectation with respect to π on both sides of (6.1), the second term involving dW on the right-hand side vanishes since it is a martingale term. By Corollary 2 and a standard argument, we have

$$\mathbb{E}_\pi \int_0^t \nabla_a f(Z(s), A(s-)) dA^c(s) = t \int \mathrm{tr}(\nabla_a f(z,a)) d\pi(z,a).$$

We conclude that, for each $t \geq 0$ and each $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$,

$$\begin{aligned} 0 \quad = \quad & t \int [T f(z,a)] d\pi(z,a) + E_\pi \int_0^t [R' \nabla_z f(Z(s), A(s-))] dY(s) \\ & + E_\pi \sum_{s \leq t} [f(Z(s), A(s)) - f(Z(s), A(s-))], \end{aligned} \tag{6.2}$$

where T is given in (4.5). This equation serves as the starting point for proving Theorem 3.

6.2 The boundary term

In this section we rewrite the boundary term in (6.2), i.e., the term involving dY. Let $\nu = (\nu_1, \ldots, \nu_J)$ be the unique vector of measures on $\partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J}$ for which

$$\int h(z,a) \nu(dz, da) = E_\pi \int_0^1 h(Z(s), A(s-)) dY(s),$$

for all continuous $h : \partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}^J$ with compact support. Note that the right-hand side is always finite as a consequence of the fact that $E_\pi Y(1) < \infty$, see (Harrison and Williams, 1987b, Section 8).

Our next goal is to give a characterization of measure ν in terms of π, which we carry out through Laplace transforms. We start with determining the support of ν.

Lemma 9. *The support of ν is $\bigcup_i (F_i \cap F_i^a)$.*

Proof. In view of Lemma 2, it is clear that A can have at most countably many jumps. For any continuous $h : \partial\mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}^J$ with compact support, we have

$$\int_0^1 h(Z(s), A(s-)) dY(s) = \int_0^1 h(Z(s), A(s)) dY(s),$$

since the measure dY is continuous and the integrand has countably many jumps by Lemma 1. It follows from the definition of ν that

$$\int h(z,a) \nu(dz, da) = E_\pi \int_0^1 h(Z(s), A(s)) dY(s). \tag{6.3}$$

The complementarity conditions $\int_0^\infty Z(t) dY(t) = 0$ and (5.4) imply the lemma. $\square$

On combining Equations (6.2) and (6.3) we obtain that for any $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} [Tf(z,a)] d\pi(z,a) + \int_{\bigcup_i (F_i \cap F_i^a)} [R' \nabla_z f(z,a)] d\nu(z,a) \\ &\quad + \frac{1}{t} E_\pi \sum_{s \le t} [f(Z(s), A(s)) - f(Z(s), A(s-))]. \end{aligned} \tag{6.4}$$

We now express the Laplace transform of ν in terms of the Laplace transform of π. Set $f(z,a) = \exp(-\eta\cdot z - \alpha\cdot a) \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J})$ where $(\eta, \alpha) \in \mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}$. After substituting f in (6.4), we obtain

$$Q(\eta,\alpha)\pi^*(\eta,\alpha) - \sum_{j=1}^J (R'\eta)_j \nu_j^*(\eta,\alpha) + H(\eta,\alpha) = 0, \tag{6.5}$$

where

$$\begin{aligned} Q(\eta,\alpha) &= \frac{1}{2}\eta'\Sigma\eta - \theta'\eta - \mathrm{tr}(\alpha), \\ \pi^*(\eta,\alpha) &= \int_{\mathbb{R}_+^J \times \mathbb{M}^{J\times J}} e^{-\eta\cdot z - \alpha\cdot a} d\pi(z,a), \\ \nu_j^*(\eta,\alpha) &= \int_{F_j \cap F_j^a} e^{-\eta\cdot z - \alpha\cdot a} d\nu_j(z,a), \\ H(\eta,\alpha) &= E_\pi \sum_{s \le 1} [e^{-\eta\cdot Z(s)} \cdot (e^{-\alpha\cdot A(s)} - e^{-\alpha\cdot A(s-)})]. \end{aligned}$$

Dividing (6.5) by $\eta_j > 0$ and letting $\eta_j \to \infty$, we deduce that

$$\nu_j^*(\eta,\alpha) = \frac{1}{2}\Sigma_{jj} \lim_{\eta_j \to \infty} \eta_j \pi^*(\eta,\alpha), \tag{6.6}$$

where we have used the fact that $\nu_j(F_j \cap F_i) = 0$ for $i \neq j$ so that $\lim_{\eta_j\to\infty} \nu_i^*(\eta,\alpha) = 0$ by the dominated convergence theorem. Under further regularity conditions on π, one can use the initial value theorem for Laplace transforms to show that $d\nu_j = \frac{1}{2}\Gamma_{jj} d\pi_j$ for an appropriate restriction π_j of π. Carrying out this procedure provides little additional insight, and we therefore suppress further details.

6.3 The jump term

We now proceed investigating the jump term, i.e., the term in (6.2) involving the countable sum. It is possible to express this term completely in terms of π using the theory of distributions; this is done in Section 7. However, to prove Theorem 3, it is possible to use (6.2) for special functions with the property that the jump term vanishes.

Throughout, we fix a set $I \subseteq \{1, \ldots, J\}$. Recall the definition of O_I in (4.2). It is our aim to show that the jump term vanishes for functions of the form $O_I f$, where $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$ as before. We first introduce a lemma.

Lemma 10. *For any $f : \mathbb{R}_+^J \times \mathbb{M}^{J\times J} \to \mathbb{R}$, if $z_j = 0$ for $j \notin I$, then for any $a \in \mathbb{M}^{J\times J}$ we have*

$$\sum_{S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) = 0.$$

In particular, if $z_j = 0$ for $j \notin I$, then we have $O_I f(z, a) = 0$.

Proof. Suppose $z_j = 0$ for some $j \notin I$. Then for any set $S\subseteq \{1, \ldots, J\}\setminus I$ with $j \notin S$, we have $\Pi_{S\cup I} z = \Pi_{S\cup I\cup\{j\}} z$. Using this observation, we deduce that

$$\begin{aligned}
&\sum_{S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a)\\
&= \sum_{j\in S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) + \sum_{j\notin S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a)\\
&= \sum_{j\in S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) + \sum_{j\notin S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I\cup\{j\}} z, a)\\
&= \sum_{j\in S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|} f(\Pi_{S\cup I} z, a) + \sum_{i\in S\subseteq\{1,\ldots,J\}\setminus I} (-1)^{|S|-1} f(\Pi_{S\cup I} z, a)\\
&= 0.
\end{aligned}$$

The proof of the lemma is complete. ☐

Now we are ready to show that the jump term vanishes for special functions $O_I f$. For any $K \subseteq \{1, \ldots, J\}$, Z_K denotes the process whose components are those of Z with indices in K.

Lemma 11. *For each $t \geq 0$ and $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}^{J\times J})$, we have*

$$E_\pi \sum_{s\leq t} [O_I f(Z(s), A(s)) - O_I f(Z(s), A(s-))] = 0. \tag{6.7}$$

Proof. By Lemma 6 and Lemma 10, we have

$$\begin{aligned}
&E_\pi \sum_{s\leq t} [O_I f(Z(s), A(s)) - O_I f(Z(s), A(s-))]\\
&= \sum_{\emptyset\neq K\subseteq\{1,\ldots,J\}} E_\pi \sum_{s\leq t: Z_K(s)=0, Z_{\{1,\ldots,J\}\setminus K}(s)>0} [O_I f(Z(s), A(s)) - O_I f(Z(s), A(s-))]\\
&= \sum_{\emptyset\neq K\subseteq I} E_\pi \sum_{s\leq t: Z_K(s)=0, Z_{\{1,\ldots,J\}\setminus K}(s)>0} [O_I f(Z(s), A(s)) - O_I f(Z(s), A(s-))].
\end{aligned}$$

Therefore, to show (6.7) it suffices to show for each nonempty set $K \subseteq I$, we have

$$E_\pi \sum_{s\le t: Z_K(s)=0, Z_{\{1,\ldots,J\}\setminus K}(s)>0} [O_I f(Z(s), A(s)) - O_I f(Z(s), A(s-))] = 0. \tag{6.8}$$

To prove (6.8) we first deduce from Lemma 7 that when $Z_K(s) = 0$ and $Z_{\{1,\ldots,J\}\setminus K}(s) > 0$,

$$Q_K(A(s-) = A(s)).$$

Next, since $K \subseteq I$, we use the projection property of the operator Q_I to obtain

$$Q_I(A(s)) = Q_I(Q_K(A(s-)) = Q_I(A(s-)).$$

Now (6.8) readily follows from and the definition of O_I as in (4.2). Thus we have completed the proof of the lemma. □

Proof of Theorem 3. Equation (4.7) immediately follows from (6.4) and Lemma 11. Summing all the equations in (4.7) over the sets $I \subseteq \{1, \ldots, J\}$, we obtain (4.6). □

7 Jump measures

In this section, we further investigate the jump term in (6.2), resulting in a characterization of jump measures in terms of the stationary distribution π. We first introduce some notation. For $I \subseteq \{1, \ldots, J\}, I \neq \emptyset$, we define measures u_I on $\mathbb{R}_+^{|I^c|} \times \mathbb{M}_+^{J\times J}$ with support in $(0,\infty)^{|I^c|} \times \mathbb{M}_+^{J\times J}$. We set, for Borel sets $B \subseteq (0,\infty)^{|I^c|}, C \subseteq \mathbb{M}_+^{J\times J}$,

$$u_I(B, C) = E_\pi \sum_{s\le 1: Z_I(s)=0, Z_{I^c}(s)\in B, A(s)\neq A(s-)} 1_C\{A(s-)\}.$$

We write z_I for the subvector of z consisting of the components with indices in I as before, and we also let $z|_I$ denote the projection of z to $\{z : z_{I^c} = 0\}$. We start with an auxiliary result on the measures u_I.

Lemma 12. *For each $I \subseteq \{1, \ldots, J\}, I \neq \emptyset$ and $k = 1, \ldots, J$, we have $u_I(\{(z_{I^c}, a) : a_k = 0\}) = 0$.*

Proof. We exploit the dynamics of the augmented Skorohod problem. Since $A_k(s-) = 0$ implies $Z_k(s) = 0$, we have $u_I(\{(z_{I^c}, a) : a_k = 0\}) = 0$ for $k \in I^c$. We next consider $k \in I$. Since the continuous part of A_k^k is strictly increasing when $Z_k > 0$, the only possibility for $Z_I(s) = 0$, $A_k(s-) = 0$, and $A(s) \neq A(s-)$ to occur simultaneously is for Z to hit the face $z_I = 0$ without leaving the face $z_k = 0$. Since the time Z spends on the boundary has Lebesgue measure zero, this cannot happen almost surely. □

To proceed with our description of the measures u_I, we need tools from theory of distributions (or generalized functions), which makes it possible to differentiate probability distributions. For background on this theory, see Duistermaat and Kolk (2010); Rudin (1991). For $I \subseteq \{1, \ldots, J\}$, we define the operator

$$T_I^* = \frac{1}{2} \sum_{i,j \in I} \Sigma_{ij} \frac{\partial^2}{\partial z_i \partial z_j} - \sum_{j \in I} \theta_j \frac{\partial}{\partial z_j} - \mathrm{tr}(\nabla a).$$

Note that, with the understanding that we use the theory of distributions, this operator can act on probability measures. We also define

$$d\pi_I(z_{I^c}, a) = \int_{z_I} d\pi(z, a).$$

The main result of this section is that u_I can be expressed in terms of π. Indeed, together with Lemma 12, it completely determines u_I.

Proposition 1. *For each $I \subseteq \{1, \ldots, J\}, I \neq \emptyset$, we have, with $z_{I^c} \in (0, \infty)^{|I^c|}, a \in \mathbb{M}_+^{J \times J}$ and $a_k \neq 0$ for $k = 1, \ldots, J$,*

$$du_I(z_{I^c}, a) = \sum_{K \subseteq I, K \neq \emptyset} (-1)^{|I \setminus K|} \int_{z_{I \setminus K}} [T_{K^c}^* d\pi_K](z_{K^c}, a).$$

Proof. The basis for the proof is (6.4). We first rewrite the jump term in (6.4) using the jump measures. In view of Lemmas 5 and 7,

$$\begin{aligned}
&E_\pi \sum_{s \leq 1} [f(Z(s), A(s)) - f(Z(s), A(s-))] \\
&= \sum_{\emptyset \neq K \subseteq \{1,\ldots,J\}} E_\pi \sum_{s \leq 1: Z_K(s)=0, Z_{K^c}(s)>0} [f(Z|_{K^c}(s), A(s)) - f(Z|_{K^c}(s), A(s-))] \\
&= \sum_{\emptyset \neq K \subseteq \{1,\ldots,J\}} E_\pi \sum_{s \leq 1: Z_K(s)=0, Z_{K^c}(s)>0, A(s) \neq A(s-)} [f(Z|_{K^c}(s), Q_K(A(s-))) - f(Z|_{K^c}(s), A(s-))] \\
&= \sum_{\emptyset \neq K \subseteq \{1,\ldots,J\}} \int_{z_{K^c}, a} [f(z|_{K^c}, Q_K(a)) - f(z|_{K^c}, a)] du_K(z_{K^c}, a).
\end{aligned}$$

Fix some nonempty $I \subseteq \{1, \ldots, J\}$. For $f \in C_b^2(\mathbb{R}_+^J \times \mathbb{M}_+^{J \times J})$ with the property that f vanishes on $\bigcup_{i \in I^c} F_i \cup \bigcup_i F_i^a$, (6.4) thus reduces to

$$\int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J \times J}} Tf(z, a) d\pi(z, a) = \sum_{I \neq L \subseteq I} \int_{z_{I^c \cup L}, a} f(z|_{I^c \cup L}, a) du_{I \setminus L}(z_{I^c \cup L}, a).$$

If moreover $f(z,a)$ does not depend on z_I, this can be simplified further:

$$\int_{\mathbb{R}_+^J \times \mathbb{M}_+^{J\times J}} Tf(z,a)d\pi(z,a) = \sum_{I\neq L\subseteq I} \int_{z_{I^c},a} f(z|_{I^c},a) \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a). \tag{7.1}$$

The left-hand side can be rewritten using the theory of differentiation for distributions (Duistermaat and Kolk, 2010, Ch. 4) or (Rudin, 1991, Sec. II.6.12). This leads to

$$\int_{\mathbb{R}_+^J \times \mathbb{M}^{J\times J}} Tf(z,a)d\pi(z,a) = \int_{z_{I^c},a} f(z|_{I^c},a)[T^*_{I^c}d\pi_I](z_{I^c},a).$$

Combining this with (7.1) and rearranging terms, we get

$$\begin{aligned}
&\int_{z_{I^c},a} f(z|_{I^c},a)du_I(z_{I^c},a) \\
&= \int_{z_{I^c},a} f(z|_{I^c},a)[T^*_{I^c}d\pi_I](z_{I^c},a) - \sum_{L\subseteq I, L\neq\emptyset, L\neq I} \int_{z_{I^c},a} f(z|_{I^c},a) \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a).
\end{aligned}$$

This shows that, for $z_{I^c} \in (0,\infty)^{|I^c|}, a \in \mathbb{M}_+^{J\times J}$ and $a_k \neq 0$ for $k = 1,\ldots,J$,

$$du_I(z_{I^c},a) = T^*_{I^c}d\pi_I(z_{I^c},a) - \sum_{L\subseteq I, L\neq\emptyset, L\neq I} \int_{z_L} du_{I\setminus L}(z_{I^c\cup L},a).$$

Since $|I\setminus L| < |I|$, this representation allows us to finish the proof of the proposition by an elementary induction argument on $|I|$. Alternatively, one could use a version of the inclusion-exclusion principle (Stanley, 1997, Sec. 2.1). □

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A Proof of (2.5)

This appendix uses Theorem 1 to find the Laplace transform of the stationary distribution π of (Z,A) in the one-dimensional case, thereby showing in particular that Theorem 1 completely determines π. Writing $\mathcal{L}(\alpha,\eta)$ for the Laplace transform of π, Theorem 1 implies that

$$\left(\frac{1}{2}\sigma^2\alpha^2 - \alpha\theta - \eta\right)\mathcal{L}(\alpha,\eta) + \eta\mathcal{L}(0,\eta) + \alpha\theta = 0. \tag{A.1}$$

In particular, on setting $\eta = \frac{1}{2}\sigma^2\alpha^2 - \alpha\theta$ we get

$$\left[\frac{1}{2}\sigma^2\alpha^2 - \alpha\theta\right]\mathcal{L}\left(0, \frac{1}{2}\sigma^2\alpha^2 - \alpha\theta\right) + \alpha\theta = 0.$$

After substitution of $\alpha = (\theta + \sqrt{\theta^2 + 2\sigma^2\eta})/\sigma^2$, we find that

$$\eta\mathcal{L}(0,\eta) = -\theta\left[\frac{\theta + \sqrt{\theta^2 + 2\sigma^2\eta}}{\sigma^2}\right].$$

Substituting this back into (A.1) and simplifying the resulting expression, we obtain the Laplace transform given in (2.5).

B The augmented Skorohod problem and uniqueness

In this appendix, we prove that the augmented Skorohod problem admits a unique solution. To this end, we employ a similar contraction map as in Lemma 3.6 of Mandelbaum and Ramanan (2010). Define a map Λ from $\mathbb{D}^{J\times J}$ to $\mathbb{D}^{J\times J}$ by setting, for $t \geq 0$,

$$\Lambda(b)_i^j(t) = \sup_{s\in\Phi_{(i)}(t)} [\chi_i^j(s) + [\tilde{P}b^j]_i(s)]. \tag{B.1}$$

Momentarily we show that Λ is a contraction map, and thus Λ has a unique fixed point b. This also implies that given $b^{(0)} = 0$, and recursively define $b^{(n+1)} = \Lambda(b^{(n)})$, we have for every $T > 0$, $||b^{(n)} - b||_T \to 0$ as $n \to \infty$. Since χ is nonnegative and nondecreasing and $\tilde{P}$ is nonnegative, we deduce that $b^{(n)}$ is componentwisely nonnegative and nondecreasing for each n. Therefore, we obtain that the fixed point b is also nonnegative and nondecreasing. Now let $a = \chi - \tilde{R}b$, $z = \Gamma(x)$, and $y = \Phi(x)$. We now verify directly that (z, y, a, b) is a solution to the augmented Skorohod problem. Only the fourth and fifth requirement in Definition 3 are not immediate. The fourth requirement can be shown to hold using the same argument as in the proof of Lemma 4. For the fifth requirement, we note that if $z_i(t) = 0$, (B.1) implies that for each j,

$$b_i^j(t) = \chi_i^j(t) + (\tilde{P}b^j)_i(t),$$

which yields

$$a_i(t) = \chi_i(t) - (\tilde{R}b)_i(t) = \chi_i(t) + (\tilde{P}b)_i(t) - b_i(t) = 0.$$

To establish the uniqueness of solutions to the augmented Skorohod problem, we use the contraction map Λ. Suppose (z, y, a, b) solves the augmented Skorohod problem. Let $\tilde{b} = \Lambda(b)$. If we can show that $\tilde{b} = b$, meaning b is a fixed point of Λ, then it follows from the uniqueness of the fixed point that there must be a unique solution to the augmented Skorohod problem. Suppose there exists some i, j and t_0 such that $\tilde{b}_i^j(t_0) \neq b_i^j(t_0)$. We

discuss two cases. If $z_i(t_0) = 0$, using nonnegativity and monotonicity of b, one can check from (B.1) that $\tilde{b}_i^j(t_0) = \chi_i^j(t_0) + [\tilde{P}b^j]_i(t_0)$. From the definition of the augmented Skorohod problem, we also know that $z_i(t_0) = 0$ implies $a_i^j(t_0) = \chi_i^j(t_0) + [\tilde{R}b^j]_i(t_0) = 0$. Therefore, we have $\tilde{b}_i^j(t_0) = b_i^j(t_0)$, a contradiction. Now consider the second case where we have $z_i(t_0) > 0$. If the set $\Phi_{(i)}(t_0)$ is empty, we have $\tilde{b}_i^j(t_0) = b_i^j(t_0) = b_i^j(0) = 0$. If not, let s be the maximal element in $\Phi_{(i)}(t_0)$. We deduce from the previous case in conjunction with the complementarily condition (4.1) that $b_i^j(t_0) = b_i^j(s) = \tilde{b}_i^j(s) = \tilde{b}_i^j(t_0)$. This is again a contradiction. Therefore, we obtain $\tilde{b} = b$ and infer that the augmented Skorohod problem has a unique solution.

It remains to show that Λ is a contraction map on $\mathbb{D}^{J\times J}$, which is equipped with the uniform norm on compact sets. As in the proof of Lemma 3.6 in Mandelbaum and Ramanan (2010) we assume that, without loss of generality, the maximum row sum of $\tilde{P}$ is $\eta < 1$. It is easy to verify that for any fixed $T > 0$,

$$||\Lambda(b) - \Lambda(b')||_T \leq \eta ||b - b'||_T$$

for all $b, b' \in \mathbb{D}^{J\times J}$. Thus we have proved the existence and uniqueness of a fixed point for Λ.

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